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ECE 204 *Numerical methods*

Frequency analysis

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Introduction

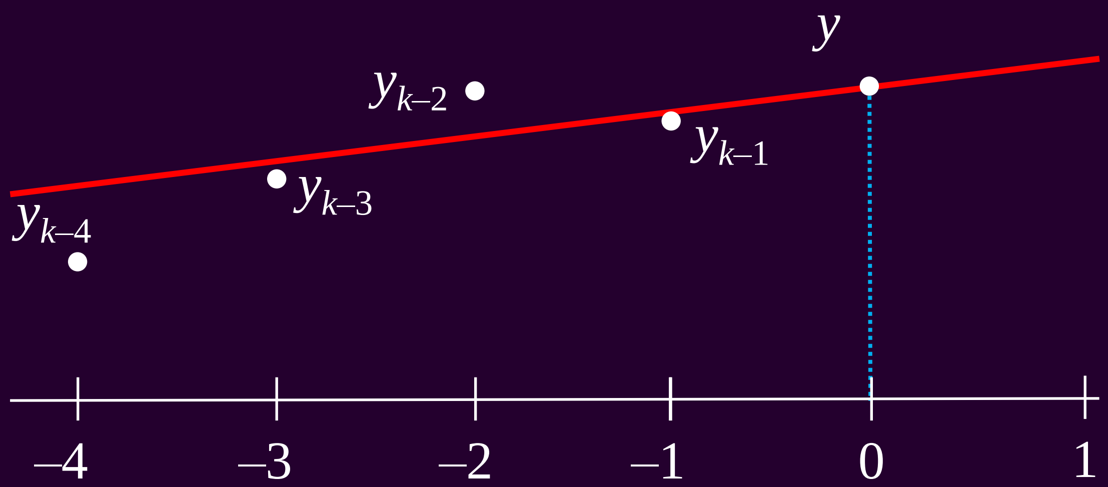
- In this topic, we will
  - Review interpolating and least-squares best-fitting polynomials
  - Describe how this is a time-domain analysis
  - Discuss how this is sub-optimal for periodic signals
  - Describe the upcoming lectures
  - Give a quick review of the properties of vectors
    - We will see that functions have these same properties

Time-domain analysis

- Up to this point, we have assumed that we are periodically sampling data in the time domain

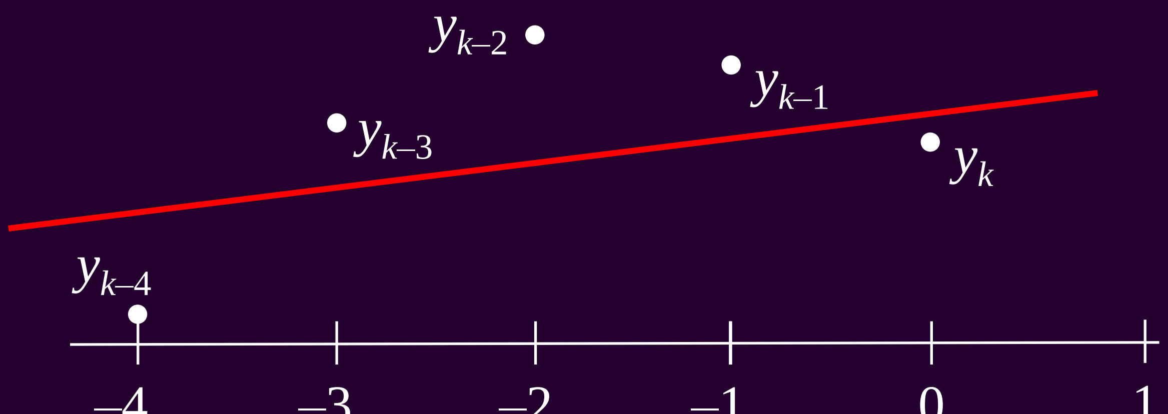
$$y = (y_0, y_1, y_2, y_3, y_4, \ldots)$$

  - Our analysis to this point has been by analyzing and finding interpolating polynomials and best-fitting least-squares polynomials that pass through these points



Periodic data

- Often, however, we are dealing with periodically varying data
  - Time-domain analysis of a periodic signal is more difficult
  - Best-fitting quadratics don't do well
  - We could find a best-fitting linear combination of sine and cosine functions but this would be expensive



Periodic data

- For example, suppose we have a periodically sampled periodic signal

$$\mathbf{t} = \begin{pmatrix} 0 \\ -1 \\ -2 \\ -3 \\ -4 \\ -5 \\ -6 \\ -7 \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} -3.2341 \\ -2.5901 \\ -2.3177 \\ -0.6899 \\ 0.1659 \\ 2.3021 \\ 2.8124 \\ 3.3246 \end{pmatrix}$$

$$y_k = a\cos\left(0.32t_k\right) + b\sin\left(0.32t_k\right)$$

$$V = \begin{pmatrix} 1 & 0 \\ 0.9492 & -0.3146 \\ 0.8021 & -0.5972 \\ 0.5735 & -0.8192 \\ 0.2867 & -0.9580 \\ -0.0292 & -0.9996 \\ -0.3421 & -0.9396 \\ -0.6204 & -0.7843 \end{pmatrix}$$

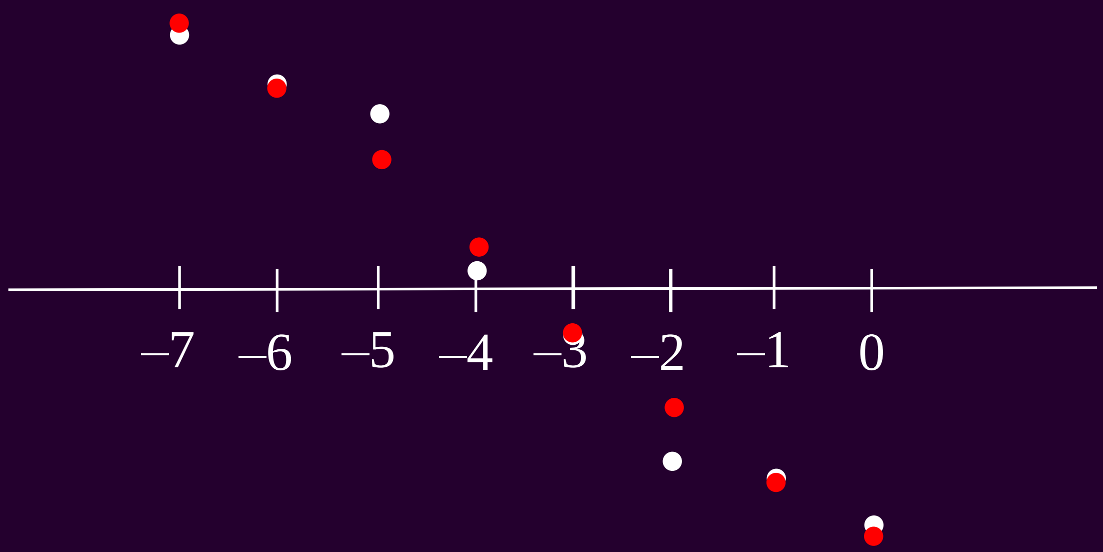
$$V^{\mathrm{T}}V\begin{pmatrix} a \\ b \end{pmatrix} = V^{\mathrm{T}}\mathbf{y}$$

$$a = -3.4985$$

$$b = -1.6166$$

Periodic data

- We can compare the noisy signal and the least-squares best-fitting sinusoid



Looking ahead

- This, however, is expensive and narrowly focused
  - Instead, we will focus on frequency analysis
  - We will transform the data into the frequencies involved

- Our approach:
  - A review of linear algebra and inner products
  - We will look at complex exponential Fourier series
  - This will motivate a discrete Fourier transform on vectors
    - We will then look at the fast Fourier transform

Finite-dimensional vectors

- Recall that the definition of a finite-dimensional vector
  - A vector in $\mathbf{R}^8$ has eight real entries
  - We can take linear combinations of these vectors

$$\mathbf{u} = \begin{pmatrix} 0.3 \\ 1.5 \\ -2.7 \\ 4.0 \\ -5.8 \\ -1.9 \\ -2.4 \\ 6.1 \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} -5.4 \\ 3.7 \\ 1.8 \\ 2.9 \\ -4.5 \\ 0.6 \\ -6.3 \\ 3.0 \end{pmatrix} \qquad 0.5\mathbf{u} + 2.1\mathbf{v} = \begin{pmatrix} -11.19 \\ 8.52 \\ 2.43 \\ 8.09 \\ -12.35 \\ 0.31 \\ -14.43 \\ 9.35 \end{pmatrix}$$

Functions are vectors

- Recall that a function of *t* is a function *f* such that we can evaluate *f* (*t*) for all *t* on a given domain
  - We can take linear combinations of these functions

$$f(t) = 3.52t^2 + 4.75t - 0.51$$

$$g(t) = -6.54t + 8.52$$

$$0.5f(t) + 2.1g(t) = 1.76t^2 - 11.359t + 17.637$$

- Conclusion?
  - Functions are vectors

Summary

- Following this topic, you now
  - Understand the analysis we have done until now is in the time domain
  - Have observed that trying to do frequency analysis is difficult
  - Understand we will, instead, look at the different frequencies that make up the signal
  - Saw a quick review of vectors, but also saw that functions share the exact same properties
    - Functions are vectors!

References

[1] https://en.wikipedia.org/wiki/Frequency_domain

Acknowledgments

None so far.

Colophon

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The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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